AMS Qualifying Exam (January 2017): Probability Questions

Solve any three of the following four problems.

All problems are weighted equally. On this cover page write which three problems you want graded.

problems to be graded:

Name (PRINT CLEARLY), ID number

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1. Let X, Y, and Z be three independent uniform random variables on $[0, 1]$. Compute the probability $P(XY < Z^2)$.

2. Let X and Y be jointly continuous with joint density function $f_{X,Y}(x, y) = \frac{1}{x}$, $0 \leq y \leq x \leq 1$. Compute the probability $P(X^2 + Y^2 \leq 1 | X = x)$.

3. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with a common cumulative distribution function F satisfying $\lim_{x\to\infty} F(x) = 1$. For a given constant l, define $Z(l) = \min\{k : X_k > l\}$. Compute $\lim_{l\to\infty} P(Z(l) \leq E[Z(l)])$.

4. Let X and Y be two continuous random variables with marginal density functions $f_X(x)$ and $f_Y(y)$. Is it true that $E[\ln f_X(X)] \geq E[\ln f_Y(X)]$? If yes, provide a proof; otherwise, give a concrete counterexample.